Notes Feb 10

I. Finite type and minimality.

The next invariant we shall discuss are Hörmander rank and type. These were introduced by Lars Hörmander in the theory of PDE to study sums of squares of vector fields.

The natural setting for this is that of abstract CR mflds.

<u>Recall</u>. A <u>CR mfld</u> is a pair $(M^{2n+d}, \mathcal{V})$ where $\mathcal{V}$ is a rank n cplx subbdle of $\mathbb{C}\otimes TM$ s.t.

<u>dim M</u>. (a) $\mathcal{V}\cap\overline{\mathcal{V}} = \{0\}$ CR dim M

(b) $[\mathcal{V},\mathcal{V}]\subseteq\mathcal{V}$. $d=$ <u>CR codim M</u>

It is sometimes convenient to translate CR structure into a "real" structure.

Let $\bar{Z}_1, \dots, \bar{Z}_n$ be a local frame for $\overline{\mathcal{V}}$ (We think of $\overline{\mathcal{V}}$ as $T^{0,1}$ so we use notation $\bar{Z}_j$), and write

$$\bar{Z}_j = X_j + i Y_j ,$$

where X_j, Y_j are real vector fields, i.e. sections of TM. We define a rank $2n$ subbundle of TM by

$$H := \left\{ \sum_{j=1}^{n} (a_j X_j + b_j Y_j) : a_j, b_j \in \mathbb{R} \right\}.$$

We define an endomorphism $J : H \to H$ by

$$J(X_j) = Y_j \,, \quad J(Y_j) = -X_j$$

Note that $J^2 = -I \Rightarrow$ eigenvalues at each p are $\pm i$, and

$$J(\overline{Z_j}) = J(X_j) + iJ(Y_j) = Y_j - iX_j = -i\overline{Z_j},$$

i.e. $\overline{\mathcal{V}}$ is the $-i$ eigenspace of J.

A frame for $\mathcal{V}$ is given by the $Z_j = X_j - iY_j$. One checks that $\mathcal{V}$ is the $+i$ eigenspace of J.

We get a triplet (M, H, J), where H rank $2n$ subbundle of TM, $J \in \mathrm{End}(H)$ s.t. $J^2 = -I$.

Conversely, given such a triplet we can let $\mathcal{V}_p, \overline{\mathcal{V}_p} \subseteq \mathbb{C} \otimes T_pM$ be $+i, -i$ eigenspaces of J at p.

The $\mathcal{V}_p$, $\overline{\mathcal{V}_p}$ then have constant (in p) $\dim = n$, and form subbdles $\mathcal{V}, \overline{\mathcal{V}}$ of $\mathbb{C} \otimes TM$. Clearly, $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$, but $[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$ requires an additional condition (vanishing Nijenhuis tensor). For simplicity, we shall just assume that J is s.t. $[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$.

<u>Levi form, Rev.</u> Assume $d=1$ (<u>hypersurface type</u>), choose real 1-form θ (near given $p \in M$) s.t. $\theta^{\perp} = \mathcal{V} \oplus \overline{\mathcal{V}}$ and let

$$\mathcal{L}_p^{\theta} : \mathcal{V} \oplus \mathcal{V} \to \mathbb{C}$$

be given by

$$\mathcal{L}_p^{\theta}(Z_p, W_p) = \theta([Z, \overline{W}])$$

Z, W are sections of $\mathcal{V}$ extending Z_p and W_p.

Ex. ① Check that $\mathcal{L}_p^\theta$ is independent of extensions Z, W (as sections of $\mathcal{V}$).

② Show that if $Z_p \in \mathcal{V}_p$ then $\exists X_p \in H_p$ s.t. $Z_p = X_p - iJX_p$

③ Show that if $Z_p, W_p \in \mathcal{V}_p$ with $Z = X_p - iJX_p$, $W = Y_p - iJY_p$

then

$$\mathcal{L}_p^\theta (Z_p, Y_p) = \frac{1}{2i}\,\theta([X, JY])$$

where X, Y are v.f. in H extending X_p, Y_p.

Hörmander dim and type.

Let $p \in M$ and $U \subseteq M$ an open nbhd of p. Let $\Gamma(U,H)$ denote sections of H in U (v.f. w/ values in H), and let $\mathfrak{g}^H_U$ denote the Lie algebra generated by $\Gamma(U,H)$, i.e. $\mathfrak{g}^H_U$ is the $\mathbb{R}$ vector space generated by $\Gamma(U,H)$ and all repeated commutators

$$[X_1,X_2],\ [[X_1,X_2],X_3],\ \ldots\ ,$$

where $X_1, X_2, X_3, \ldots \in \Gamma(U,H)$

__Def.__ We say $[X_1,X_2]$ has __length__ 2, $[[X_1,X_2],X_3]$ __length__ 3, etc.

We'll write $\mathfrak{g} = \mathfrak{g}_U^H$ for brevity and note that $\mathfrak{g}_p \subseteq T_pM \Rightarrow \exists\, m$ s.t. $\mathfrak{g}_p$ is spanned by commutators of length $\leq m$ evaluated at p.

<u>Def</u>. ① The Hörmander <u>rank</u> and <u>type</u> at p are $\dim \mathfrak{g}_p$ and m, respectively.

② M is of (Hörmander) <u>finite type</u> at p if $\dim \mathfrak{g}_p = \dim M$.

<u>Prop</u>. If $M' \subseteq M$ is a smooth submfld, with $p \in M'$ and $H \subseteq TM'$, then $\dim M' \geq \dim \mathfrak{g}_p$, where $\mathfrak{g} = \mathfrak{g}_U^H$.

Pf. Obvious, since $H \subseteq TM' \Rightarrow$ any repeated commutator $[\cdots[X_1, X_2], X_3] \ldots]$ is tangent to M' as well $\Rightarrow \mathfrak{g}_p \subseteq T_p M'$ ☒

<u>Def 2</u>. M is <u>minimal</u> at p if $\nexists$ $p \in M' \subseteq M$ s.t. $H \subseteq TM'$ and $\dim M' < \dim M$.

<u>Cor 1</u>. If M is of finite type at p, then M is minimal at p.

<u>Rem</u>. Converse is <u>not</u> true in general but is true if M is real analytic by Nagano's Thm that we will discuss later.

The "opposite" of finite type occurs if $\dim \mathfrak{g}_p = 2n = \dim H_p$.

Ex. Assume H.-rank $= 2n$, $\forall q \in U$.
Then H is integrable (Frobenius condition). $\Rightarrow$ by Frobenius $\Rightarrow$ U is foliated by submanifolds M' s.t. $H = TM'$. Then J gives an almost complex structure on TM'.
Since we assume that J is s.t.
$[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$ ($\Rightarrow N(J) = 0$)
Nijenhuis

Newlander-Nirenberg Thm $\Rightarrow$ (M', J) are cplx (holom.) mflds.

IKASCV

Formulation in BER (Thm 2.1.26). Pf in e.g. Hörmander